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in multiobjective programming

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Generating epsilon-efficient solutions in multiobjective programming

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Abstract

Scalarization approaches to purposely generating epsilon-efficient solutions of multiobjective programs are investigated and a generic procedure for computing these solutions is proposed and illustrated with an example. Real-life decision making situations in which the solutions are of significance are described.

Keywords: multiobjective programs, ϵ -efficient solutions, ϵ -Pareto outcomes, ϵ -nondominated outcomes, scalarizations, ϵ -optimality.

1 Introduction

In the middle of the nineteen eighties, Loridan (1984) introduced a notion of ϵ -efficient solutions for multiobjective programs (MOPs), which was followed by White (1986) who proposed several concepts of approximate solutions for MOPs and drafted methods for their generation.

For the last two decades, ϵ -efficient or approximate solutions of MOPs have been examined in the literature by many authors from different points of view. Existence conditions were developed by Deng (1997) and Dutta and Vetrivel (2001) for convex MOPs while KKT-type conditions and saddle point conditions were derived by Dutta and Vetrivel (2001) and Liu (1996). The latter also proposed ϵ -properly-efficient solutions for convex non-differentiable MOPs. Connections between different definitions of approximate solutions were analyzed by Yokoyama (1996, 1999). Vályi (1985), Tammer (1994), Tanaka (1996), and others studied approximate solutions of vector optimization problems in general ordered vector spaces.

Approximate efficient solutions have also been used for approximation of the Pareto set of specially structured biobjective programs (BOPs). In this vein, Ruhe and Fruhwirth (1990) developed an approximation algorithm for a minimum cost flow problem, Blanquero and Carrizosa (2002)

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worked with a location problem, and Angel et al. (2003) studied a scheduling problem. Furthermore, White (1998b) applied the concept of ε -efficient solutions to a portfolio selection problem, and to a linear MOP with a specially structured feasible set (White (1998a)). Fadel et al. (2002) attempted to examine the curvature of the Pareto curve for BOPs using ε -efficiency as a measure of sensitivity.

In view of the literature, the current belief is that the concept of ε -efficient solutions accounts for modeling limitations or computational inaccuracies, and thus is tolerable rather than desirable. Consequently, methods purposely avoiding efficiency and guaranteeing ε -efficiency have not been well developed.

The objective of this paper is to propose methods for generation of ε -efficient solutions. In Section 2, we formulate the problem of interest and present the terminology used in this paper. In Section 3, we explore relationships between MOPs and associated single objective programs (SOPs), a subject matter that has earlier been undertaken by some authors. References specific for each of the SOPs are listed in the subsections of Section 3 in which these SOPs are examined.

White (1986, 1998b) seems to be the only one to have addressed the practical issue of computing ε -efficient solutions. The scheme he advocates consists in the computation of approximate efficient solutions by means of approximate optimal solutions of an SOP associated with the original MOP. In Section 4, we follow on this scheme and offer specific guidance relating approximate optimality to approximate efficiency. In Section 5, we present an illustrative example and discuss real-life decision making situations that motivate this study of approximate efficiency. Section 6 concludes the paper.

2 Terminology and Problem Formulation

We first introduce some basic notation. Let \mathbb{R}^n and \mathbb{R}^m be finite dimensional Euclidean vector spaces and $y, y' \in \mathbb{R}^m$. $y > y'$ denotes $y_i > y'_i$ for all $i = 1, \dots, m$. $y \geq y'$ denotes $y_i \geq y'_i$ for all $i = 1, \dots, m$. $y \geq y'$ denotes $y \geq y'$ but $y \neq y'$. The relations \leq , \leq and $<$ are defined in the obvious way. Let $\mathbb{R}_{\geq}^m := \{y \in \mathbb{R}^m : y \geq 0\}$. The sets $\mathbb{R}_{\geq}^m, \mathbb{R}_{>}^m$ are defined accordingly.

Let $X \subset \mathbb{R}^n$ be a feasible set of decisions and let f be a vector-valued objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ composed of m real-valued functions, $f = (f_1, \dots, f_m)$, where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 1, \dots, m$. The multiobjective program (MOP) is given by

$$\text{MOP: minimize } (f_1(x), \dots, f_m(x)) \text{ subject to } x \in X$$

where the minimization is understood as finding the set of efficient solutions in X .

Definition 2.1 *Consider the MOP. A point $\hat{x} \in X$ is called*

- (i) *a weakly efficient solution if there does not exist $x \in X$ such that $f(x) < f(\hat{x})$;*

(ii) an efficient solution if there does not exist $x \in X$ such that $f(x) \leq f(\hat{x})$.

A feasible solution (decision) $x \in X$ is evaluated by the m objective functions producing the outcome $f(x)$. We define the set of all attainable outcomes or criterion vectors for all feasible solutions in the objective space, $Y := f(X) \subset \mathbb{R}^m$. The image $f(x) \in Y$ of a (weakly) efficient point is called a (weak) Pareto outcome.

Yu (1974) introduced the concept of nondominated outcomes as a generalization of Pareto outcomes.

Definition 2.2 Let $D \subset \mathbb{R}^m$ be a cone and $Y \subset \mathbb{R}^m$. Then $\hat{y} \in Y$ is called

- (i) a nondominated outcome of the MOP if there does not exist $y \in Y, y \neq \hat{y}$ and $d \in D$ such that $\hat{y} = y + d$, or equivalently, $(\hat{y} - D \setminus \{0\}) \cap Y = \emptyset$;
- (ii) a weakly nondominated outcome of the MOP if $(\hat{y} - \text{int } D) \cap Y = \emptyset$.

For $D = \mathbb{R}_{\geq}^m$, the nondominated outcomes become the Pareto outcomes. Following Loridan (1984) we define ε -efficient solutions of the MOP.

Definition 2.3 Consider the MOP and let $\varepsilon \in \mathbb{R}_{\geq}^m$. A point $\hat{x} \in X$ is called

- (i) a weakly ε -efficient solution if there does not exist $x \in X$ such that $f(x) < f(\hat{x}) - \varepsilon$;
- (ii) an ε -efficient solution if there does not exist $x \in X$ such that $f(x) \leq f(\hat{x}) - \varepsilon$.

The image $f(x) \in Y$ of a (weakly) ε -efficient point is called a (weak) ε -Pareto outcome. Following Yu (1974) we define (weakly) ε -nondominated outcomes.

Definition 2.4 Let $D \subset \mathbb{R}^m$ be a cone, $\varepsilon \in D$, and $Y \subset \mathbb{R}^m$. Then $\hat{y} \in Y$ is called

- (i) an ε -nondominated outcome of the MOP if there does not exist $y \in Y, y \neq \hat{y}$ and $d \in D$ such that $\hat{y} = y + d + \varepsilon$, or equivalently, $(\hat{y} - (D + \varepsilon) \setminus \{0\}) \cap Y = \emptyset$;
- (ii) a weakly ε -nondominated outcome of the MOP if $(\hat{y} - \text{int}(D + \varepsilon)) \cap Y = \emptyset$.

Given the MOP, we formulate a scalarized (single objective) program (SOP). Let $S \subseteq X$ be a subset of the decision set X , U be a set of auxiliary variables, and Π be a set of parameters. Let $T := f(S)$ denote the set of attainable outcomes for the SOP and $s : T \times U \times \Pi \rightarrow \mathbb{R}$ be a scalarizing function. Then the SOP associated with the MOP is given by

$$\text{SOP}(\pi): \text{minimize } s(f(x), u, \pi) \text{ subject to } x \in S, u \in U$$

where $\pi \in \Pi$ is a vector of parameters chosen by a decision maker. We next review the notion of optimality and ϵ -optimality for the SOP.

Definition 2.5 Consider the SOP and let $\epsilon \geq 0$. A point $(\hat{x}, \hat{u}) \in S \times U$ is called

- (i) an optimal solution if $s(f(\hat{x}), \hat{u}, \pi) \leq s(f(x), u, \pi)$ for all $(x, u) \in S \times U$; the outcome $\hat{y} = f(\hat{x}) \in T$ is called optimal;
- (ii) strictly ϵ -optimal solution if $s(f(\hat{x}), \hat{u}, \pi) < s(f(x), u, \pi) + \epsilon$ for all $(x, u) \in S \times U$; the outcome $\hat{y} = f(\hat{x}) \in T$ is called strictly ϵ -optimal;
- (iii) ϵ -optimal solution if $s(f(\hat{x}), \hat{u}, \pi) \leq s(f(x), u, \pi) + \epsilon$ for all $(x, u) \in S \times U$; the outcome $\hat{y} = f(\hat{x}) \in T$ is called ϵ -optimal.

Note that by definition, each optimal solution of the SOP is in particular strictly ϵ -optimal, and each strictly ϵ -optimal solution is also ϵ -optimal.

3 ϵ -Optimality and ε -Efficiency

In this section, we review seven well-known scalarization methods for the MOP and establish the corresponding relationships between $\epsilon \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}^m$ when solving both the scalarized problem and the MOP for ϵ -optimality and ε -efficiency, respectively.

3.1 Weighted-sum scalarization

The weighted-sum method for multiobjective programming is defined as

$$\text{WS}(w): \text{minimize } \sum_{i=1}^m w_i f_i(x) \text{ subject to } x \in X,$$

where $w \in \mathbb{R}_{\geq}^m$ is a given weighting parameter. Observe that for this scalarization, the feasible set is given by $S = X$ so that $T = Y$ is the complete set of outcomes. Moreover, $U = \emptyset$ and $\Pi = \mathbb{R}_{\geq}^m$.

White (1986), Deng (1997) and Dutta and Vetrivel (2001) studied this method in the context of relationships between ϵ -optimality for the SOP and ε -efficiency for the MOP. Following on their work, we develop more specific results.

Proposition 3.1 Given the MOP and $\varepsilon \in \mathbb{R}^m$, let $\epsilon \leq \sum_{i=1}^m w_i \varepsilon_i$.

- (i) If $\hat{y} \in Y$ is strictly ϵ -optimal for the $\text{WS}(w)$, then \hat{y} is ε -Pareto for the MOP.
- (ii) If $\hat{y} \in Y$ is ϵ -optimal for the $\text{WS}(w)$, then \hat{y} is weak ε -Pareto for the MOP.
- (iii) If $\hat{y} \in Y$ is ϵ -optimal for the $\text{WS}(w)$ with $w \in \mathbb{R}_{>}^m$, then \hat{y} is ε -Pareto for the MOP.

Proof Although quite similar, we give all three proofs in detail.

For (i), let $\hat{y} \in Y$ be strictly ϵ -optimal for the $WS(w)$,

$$\sum_{i=1}^m w_i \hat{y}_i < \sum_{i=1}^m w_i y_i + \epsilon \text{ for all } y \in Y, y \neq \hat{y},$$

and suppose that \hat{y} is not ϵ -Pareto. Then there exists $y \in Y, y \neq \hat{y}$ such that $\hat{y} \geq y + \epsilon$, yielding

$$\sum_{i=1}^m w_i \hat{y}_i \geq \sum_{i=1}^m w_i y_i + \sum_{i=1}^m w_i \epsilon_i \geq \sum_{i=1}^m w_i y_i + \epsilon$$

in contradiction to the above. Hence, \hat{y} is ϵ -Pareto for the MOP.

For (ii), let $\hat{y} \in Y$ be ϵ -optimal for the $WS(w)$,

$$\sum_{i=1}^m w_i \hat{y}_i \leq \sum_{i=1}^m w_i y_i + \epsilon \text{ for all } y \in Y, y \neq \hat{y},$$

and suppose that \hat{y} is not weak ϵ -Pareto. Then there exists $y \in Y, y \neq \hat{y}$ such that $\hat{y} > y + \epsilon$, yielding

$$\sum_{i=1}^m w_i \hat{y}_i > \sum_{i=1}^m w_i y_i + \sum_{i=1}^m w_i \epsilon_i \geq \sum_{i=1}^m w_i y_i + \epsilon$$

in contradiction to the above. Hence, \hat{y} is weak ϵ -Pareto for the MOP.

For (iii), let $\hat{y} \in Y$ be ϵ -optimal for the $WS(w)$ with $w \in \mathbb{R}_{>}^m$,

$$\sum_{i=1}^m w_i \hat{y}_i \leq \sum_{i=1}^m w_i y_i + \epsilon \text{ for all } y \in Y, y \neq \hat{y},$$

and suppose that \hat{y} is not ϵ -Pareto. Then there exists $y \in Y, y \neq \hat{y}$ such that $\hat{y} \geq y + \epsilon$, yielding

$$\sum_{i=1}^m w_i \hat{y}_i > \sum_{i=1}^m w_i y_i + \sum_{i=1}^m w_i \epsilon_i \geq \sum_{i=1}^m w_i y_i + \epsilon$$

in contradiction to the above. Hence, \hat{y} is ϵ -Pareto for the MOP. □

Corollary 3.1 *Given the MOP and $\epsilon \in \mathbb{R}^m$, let $\epsilon \leq \sum_{i=1}^m w_i \epsilon_i$.*

- (i) *If $\hat{x} \in X$ is strictly ϵ -optimal for the $WS(w)$, then \hat{x} is ϵ -efficient for the MOP.*
- (ii) *If $\hat{x} \in X$ is ϵ -optimal for the $WS(w)$, then \hat{x} is weakly ϵ -efficient for the MOP.*
- (iii) *If $\hat{x} \in X$ is ϵ -optimal for the $WS(w)$ with $w \in \mathbb{R}_{>}^m$, then \hat{x} is ϵ -efficient for the MOP.*

3.2 Constrained-objective scalarization

The constrained-objective scalarization of the MOP, also known as the epsilon-constraint method (see Chankong and Haimes (1983)), is defined as

$$\text{CO}(\delta): \text{minimize } f_1(x) \text{ subject to } x \in S := \{x \in X : f_i(x) \leq \delta_i, i = 2, \dots, m\},$$

where $\delta_i \in \mathbb{R}, i = 2, \dots, m$ are given upper bounds on the objectives $f_i, i = 2, \dots, m$. Observe that for this scalarization $T = \{y \in Y : y_i \leq \delta_i, i = 2, \dots, m\}$, $U = \emptyset$ and $\Pi = \mathbb{R}^{m-1}$.

Relationships between ϵ -optimal solutions of this scalarization and ε -efficient decisions of the MOP were also examined by White (1986).

Proposition 3.2 *Given the MOP and $\varepsilon \in \mathbb{R}_{\geq}^m$, let $\epsilon \leq \varepsilon_1$.*

(i) *If $\hat{y} \in T$ is strictly ϵ -optimal for the $\text{CO}(\delta)$, then \hat{y} is ε -Pareto for the MOP.*

(ii) *If $\hat{y} \in T$ is ϵ -optimal for the $\text{CO}(\delta)$, then \hat{y} is weak ε -Pareto for the MOP.*

Proof We only prove (i) and then (ii) follows analogously. Let $\hat{y} \in T$ be strictly ϵ -optimal for the $\text{CO}(\delta)$,

$$\hat{y}_1 < y_1 + \epsilon \text{ for all } y \in T, y \neq \hat{y},$$

and suppose that \hat{y} is not ε -Pareto. Then there exists $y \in Y, y \neq \hat{y}$ such that $\hat{y} \geq y + \varepsilon$, yielding

$$\hat{y}_1 \geq y_1 + \varepsilon_1 \geq y_1 + \epsilon;$$

$$\hat{y}_i \geq y_i + \varepsilon_i \geq y_i, \quad i = 2, \dots, m.$$

Since $\hat{y} \in T$, we have $\hat{y}_i \leq \delta_i, i = 2, \dots, m$, and therefore $y \in T$ in contradiction to the above. Hence, \hat{y} is ε -Pareto for the MOP. \square

Corollary 3.2 *Given the MOP and $\varepsilon \in \mathbb{R}_{\geq}^m$, let $\epsilon \leq \varepsilon_1$.*

(i) *If $\hat{x} \in S$ is strictly ϵ -optimal for the $\text{CO}(\delta)$, then \hat{x} is ε -efficient for the MOP.*

(ii) *If $\hat{x} \in S$ is ϵ -optimal for the $\text{CO}(\delta)$, then \hat{x} is weakly ε -efficient for the MOP.*

3.3 Guddat scalarization

This scalarization developed by Guddat et al. (1985) combines the weighted-sum with the constrained-objective scalarization and is defined as

$$\text{G}(x^\circ, w): \text{minimize } \sum_{i=1}^m w_i f_i(x) \text{ subject to } x \in S := \{x \in X : f(x) \leq f(x^\circ)\},$$

where $x^\circ \in X$ is a given feasible decision and $w \in \mathbb{R}_{\geq}^m$ is a given weighting parameter.

Denote $y^\circ = f(x^\circ)$ and observe that for this scalarization $T = \{y \in Y : y \leq y^\circ\}$, $U = \emptyset$ and $\Pi = X \times \mathbb{R}_{\geq}^m$.

Proposition 3.3 *Given the MOP and $\varepsilon \in \mathbb{R}_{\geq}^m$, let $\epsilon \leq \sum_{i=1}^m w_i \varepsilon_i$.*

- (i) *If $\hat{y} \in T$ is strictly ϵ -optimal for the $G(x^\circ, w)$, then \hat{y} is ε -Pareto for the MOP.*
- (ii) *If $\hat{y} \in T$ is ϵ -optimal for the $G(x^\circ, w)$, then \hat{y} is weak ε -Pareto for the MOP.*
- (iii) *If $\hat{y} \in T$ is ϵ -optimal for the $G(x^\circ, w)$ with $w \in \mathbb{R}_{>}^m$, then \hat{y} is ε -Pareto for the MOP.*

Proof In principle, each part is proven as shown for the weighted-sum in Proposition 3.1. However, due to the additional constraints in the Guddat scalarization, we have to verify that the attainable outcome $y \in Y$ yielding the contradiction in each of the three cases is also attainable in the Guddat formulation, i.e., $y \in T$. We only give the additional details for (i) and then (ii) and (iii) follow analogously.

Let $\hat{y} \in T$ be strictly ϵ -optimal for the $G(x^\circ, w)$, and suppose that \hat{y} is not ε -Pareto. Then there exists $y \in Y, y \neq \hat{y}$ with $\hat{y} \geq y + \varepsilon$, yielding

$$y \leq \hat{y} - \varepsilon \leq \hat{y} \leq y^\circ$$

since $\hat{y} \in T$ is attainable for $G(x^\circ, w)$. This verifies $y \in T$, and the proof is complete. \square

Corollary 3.3 *Given the MOP and $\varepsilon \in \mathbb{R}_{\geq}^m$, let $\epsilon \leq \sum_{i=1}^m w_i \varepsilon_i$.*

- (i) *If $\hat{x} \in S$ is strictly ϵ -optimal for the $G(x^\circ, w)$, then \hat{x} is ε -efficient for the MOP.*
- (ii) *If $\hat{x} \in S$ is ϵ -optimal for the $G(x^\circ, w)$, then \hat{x} is weakly ε -efficient for the MOP.*
- (iii) *If $\hat{x} \in S$ is ϵ -optimal for the $G(x^\circ, w)$ with $w \in \mathbb{R}_{>}^m$, then \hat{x} is ε -efficient for the MOP.*

3.4 Benson scalarization

This scalarization developed by Benson (1978) is defined as

$$B(x^\circ): \text{ minimize } \sum_{i=1}^m l_i \text{ subject to } f(x^\circ) + l \in f(X) = Y, l \leq 0,$$

where $x^\circ \in X$ is a given feasible decision.

Observe that the Benson scalarization uses $U = \{l \in \mathbb{R}^m : l \leq 0\} = -\mathbb{R}_{\geq}^m$ as set of auxiliary variables. Furthermore, for this scalarization $S = \{x \in X : f(x) \leq f(x^\circ)\}$, $T = \{y \in Y : y \leq f(x^\circ)\}$ and $\Pi = X$.

The above formulation of the Benson scalarization can be rewritten as

$$\text{minimize } \sum_{i=1}^m f_i(x) - f_i(x^\circ) \text{ subject to } x \in S,$$

where $\min_{x \in S} \{\sum_{i=1}^m f_i(x) - f_i(x^\circ)\} = \min_{x \in S} \{\sum_{i=1}^m f_i(x)\} - \sum_{i=1}^m f_i(x^\circ)$. We conclude that

$$\tilde{B}(x^\circ) : \text{minimize } \sum_{i=1}^m f_i(x) \text{ subject to } x \in S := \{x \in X : f(x) \leq f(x^\circ)\}$$

is equivalent to the Benson problem $B(x^\circ)$, and therefore that $B(x^\circ) \equiv \tilde{B}(x^\circ) \equiv G(x^\circ, 1)$.

Therefore, we can immediately derive that Proposition 3.3(iii) and Corollary 3.3(iii) also hold true for the Benson scalarization. Nevertheless, we present an alternative proof based on the original problem formulation by Benson.

Proposition 3.4 *Given the MOP and $\varepsilon \in \mathbb{R}_{\geq}^m$, let $\epsilon \leq \sum_{i=1}^m \varepsilon_i$.*

(i) *If $\hat{y} = y^\circ + \hat{l} \in Y$ is ϵ -optimal for the $B(x^\circ)$, then \hat{y} is ε -Pareto for the MOP.*

(ii) *If in addition $\sum_{i=1}^m \hat{l}_i = 0$, then $y^\circ = f(x^\circ) \in Y$ is ε -Pareto for the MOP.*

Proof For (i), let $\hat{y} = y^\circ + \hat{l} \in Y$ be ϵ -optimal for the $B(x^\circ)$ with $y^\circ = f(x^\circ)$,

$$\sum_{i=1}^m \hat{l}_i \leq \sum_{i=1}^m l_i + \epsilon \text{ whenever } y^\circ + l \in Y, l \leq 0,$$

and suppose that \hat{y} is not ε -nondominated. Then there exists $y = y^\circ + l \in Y, y \neq \hat{y}$ such that $\hat{y} \geq y + \varepsilon$, yielding

$$l = y - y^\circ \leq \hat{y} - \varepsilon - y^\circ = \hat{l} - \varepsilon \leq 0$$

and thus

$$\sum_{i=1}^m \hat{l}_i > \sum_{i=1}^m l_i + \sum_{i=1}^m \varepsilon_i \geq \sum_{i=1}^m l_i + \epsilon$$

in contradiction to the above. Hence, \hat{y} is ε -Pareto for the MOP.

The second statement now follows from the first by observing that $\sum_{i=1}^m \hat{l}_i = 0$ with $\hat{l} \leq 0$ implies that $\hat{l} = 0$, and thus $\hat{y} = y^\circ = f(x^\circ)$. \square

Corollary 3.4 *Given the MOP and $\varepsilon \in \mathbb{R}_{\geq}^m$, let $\epsilon \leq \sum_{i=1}^m \varepsilon_i$.*

(i) *If $\hat{x} \in X$ is ϵ -optimal for the $B(x^\circ)$, then \hat{x} is ε -efficient for the MOP.*

(ii) *If in addition $\sum_{i=1}^m \hat{l}_i = 0$, then x° is ε -efficient for the MOP.*

3.5 Min-max scalarization

The min-max scalarization, also known as the max-ordering approach (see Kouvelis and Yu (1997)), is defined as

$$\text{MM: minimize } \max_{i=1,\dots,m} \{f_i(x)\} \text{ subject to } x \in X.$$

Observe that for this scalarization $S = X$, $Y = T$ and both $U = \Pi = \emptyset$.

Proposition 3.5 *Given the MOP and $\varepsilon \in \mathbb{R}_{\geq}^m$, let $\epsilon \leq \min_{i=1,\dots,m} \{\varepsilon_i\}$.*

(i) *If $\hat{y} \in Y$ is strictly ϵ -optimal for the MM, then \hat{y} is ε -Pareto for the MOP.*

(ii) *If $\hat{y} \in Y$ is ϵ -optimal for the MM, then \hat{y} is weak ε -Pareto for the MOP.*

Proof We only prove (i) and then (ii) follows analogously. Let $\hat{y} \in Y$ be strictly ϵ -optimal for the MM,

$$\max_{i=1,\dots,m} \{\hat{y}_i\} < \max_{i=1,\dots,m} \{y_i\} + \epsilon \text{ for all } y \in Y, y \neq \hat{y},$$

and suppose that \hat{y} is not ε -Pareto. Then there exists $y \in Y, y \neq \hat{y}$ such that $\hat{y} \geq y + \varepsilon$, yielding

$$\begin{aligned} \max_{i=1,\dots,m} \{\hat{y}_i\} &\geq \max_{i=1,\dots,m} \{y_i + \varepsilon_i\} \\ &\geq \max_{i=1,\dots,m} \{y_i\} + \min_{i=1,\dots,m} \{\varepsilon_i\} \\ &\geq \max_{i=1,\dots,m} \{y_i\} + \epsilon \end{aligned}$$

in contradiction to the above. Hence, \hat{y} is ε -Pareto for the MOP. □

Corollary 3.5 *Given the MOP and $\varepsilon \in \mathbb{R}^m$, let $\epsilon \leq \min_{i=1,\dots,m} \{\varepsilon_i\}$.*

(i) *If $\hat{x} \in X$ is strictly ϵ -optimal for the MM, then \hat{x} is ε -efficient for the MOP.*

(ii) *If $\hat{x} \in X$ is ϵ -optimal for the MM, then \hat{x} is weakly ε -efficient for the MOP.*

3.6 Tchebycheff-norm scalarization

The weighted Tchebycheff-norm scalarization of the MOP is defined as

$$\text{TN}(r, w): \text{ minimize } \max_{i=1,\dots,m} \{w_i(f_i(x) - r_i)\} \text{ subject to } x \in X,$$

where $r \in \mathbb{R}^m$ is a given reference or utopia point (see Steuer (1986)) and $w \in \mathbb{R}_{\geq}^m$ is a given weighting parameter.

Observe that for this scalarization $S = X$, $T = Y$, $U = \emptyset$ and $\Pi = \mathbb{R}^m \times \mathbb{R}_{\geq}^m$.

Moreover, note that the min-max scalarization discussed previously is a special case of the weighted Tchebycheff-norm formulation. More precisely, if we choose the reference point $r = 0 \in \mathbb{R}^m$ and the weighting parameter $w = 1 \in \mathbb{R}_{>}^m$, then $\text{MM} \equiv \text{TN}(0,1)$.

Proposition 3.6 *Given the MOP and $\varepsilon \in \mathbb{R}_{\geq}^m$, let $\epsilon \leq \min_{i=1,\dots,m} \{w_i \varepsilon_i\}$.*

- (i) *If $\hat{y} \in Y$ is strictly ϵ -optimal for the $\text{TN}(r, w)$, then \hat{y} is ε -Pareto for the MOP.*
- (ii) *If $\hat{y} \in Y$ is ϵ -optimal for the $\text{TN}(r, w)$, then \hat{y} is weak ε -Pareto for the MOP.*

Proof We only prove (i) and then (ii) follows analogously. Let $\hat{y} \in Y$ be strictly ϵ -optimal for the $\text{TN}(r, w)$,

$$\max_{i=1,\dots,m} \{w_i(\hat{y}_i - r_i)\} < \max_{i=1,\dots,m} \{w_i(y_i - r_i)\} + \epsilon \text{ for all } y \in Y, y \neq \hat{y},$$

and suppose that \hat{y} is not ε -Pareto. Then there exists $y \in Y, y \neq \hat{y}$ such that $\hat{y} \geq y + \varepsilon$, yielding

$$\begin{aligned} \max_{i=1,\dots,m} \{w_i(\hat{y}_i - r_i)\} &\geq \max_{i=1,\dots,m} \{w_i(y_i - r_i + \varepsilon_i)\} \\ &\geq \max_{i=1,\dots,m} \{w_i(y_i - r_i)\} + \min_{i=1,\dots,m} \{w_i \varepsilon_i\} \\ &\geq \max_{i=1,\dots,m} \{w_i(y_i - r_i)\} + \epsilon \end{aligned}$$

in contradiction to the above. Hence, \hat{y} is ε -Pareto for the MOP. □

Corollary 3.6 *Given the MOP, let $\epsilon \leq \min_{i=1,\dots,m} \{w_i \varepsilon_i\}$.*

- (i) *If $\hat{x} \in X$ is strictly ϵ -optimal for the $\text{TN}(r, w)$, then \hat{x} is ε -efficient for the MOP.*
- (ii) *If $\hat{x} \in X$ is ϵ -optimal for the $\text{TN}(r, w)$, then \hat{x} is weakly ε -efficient for the MOP.*

3.7 Pascoletti-Serafini scalarization

The direction method developed by Pascoletti and Serafini (1984) for the MOP is defined as

$$\text{PS}(r, v): \text{minimize } \mu \text{ subject to } r + \mu v - d \in f(X) = Y, d \in D,$$

where $r \in \mathbb{R}^m$ is a given reference point and $v \in D \setminus \{0\}$ is a given dominated direction. Here $D \subset \mathbb{R}^m$ is a convex cone implying that $D + D \subseteq D$.

Observe that this scalarization uses $\mu \in \mathbb{R}$ as an auxiliary variable, $d \in C$ is a dominated slack vector and $\Pi = \mathbb{R}^m \times D \setminus \{0\}$.

Proposition 3.7 *Given the MOP for a convex cone $D \subset \mathbb{R}^m$ and $\varepsilon \in D$, let $\epsilon \in \{\delta : \varepsilon - \delta v \in D\}$ and denote $d_\epsilon = \varepsilon - \epsilon v \in D$.*

(i) If $\hat{y} = r + \hat{\mu}v - \hat{d} \in Y$ is strictly ϵ -optimal for the $PS(r, v)$, then \hat{y} is ϵ -nondominated for the MOP.

(ii) If \hat{y} is ϵ -optimal for the $PS(r, v)$, then \hat{y} is weakly ϵ -nondominated for the MOP.

Proof We give the complete proof for both (i) and (ii). For (i), let \hat{y} with $\hat{d} \in D$ be strictly ϵ -optimal for the $PS(r, v)$,

$$\hat{\mu} < \mu + \epsilon \text{ whenever } r + \mu v - d \in Y \text{ for some } d \in D,$$

and suppose that \hat{y} is not ϵ -nondominated for the MOP. Then there exists $y \in Y, y \neq \hat{y}$ and $d \in D$ such that $\hat{y} = y + d + \epsilon$, yielding

$$\begin{aligned} y &= r + \hat{\mu}v - \hat{d} - d - \epsilon \\ &= r + \hat{\mu}v - \hat{d} - d - (\epsilon v + d_\epsilon) \\ &= r + (\hat{\mu} - \epsilon)v - (\hat{d} + d + d_\epsilon), \end{aligned}$$

where $\hat{d} + d + d_\epsilon \in D$ as D is a convex cone. Now denote $\mu = \hat{\mu} - \epsilon$, and then $\hat{\mu} = \mu + \epsilon$ in contradiction to the above. Hence, \hat{y} is ϵ -nondominated for the MOP.

For (ii), let \hat{y} with $\hat{d} \in D$ be ϵ -optimal for the $PS(r, v)$,

$$\hat{\mu} \leq \mu + \epsilon \text{ whenever } r + \mu v - d \in Y \text{ for some } d \in D,$$

and suppose that \hat{y} is not weakly ϵ -nondominated for the MOP. Then there exists $y \in Y, y \neq \hat{y}, d \in \text{int } D$ such that $\hat{y} = y + d + \epsilon$. Since $d \in \text{int } D$, there also exists $\gamma > 0$ such that $d_\gamma = d - \gamma v \in D$ yielding

$$\begin{aligned} y &= r + \hat{\mu}v - \hat{d} - d - \epsilon \\ &= r + \hat{\mu}v - \hat{d} - (\gamma v + d_\gamma) - (\epsilon v + d_\epsilon) \\ &= r + (\hat{\mu} - \gamma - \epsilon)v - (\hat{d} + d_\gamma + d_\epsilon), \end{aligned}$$

where $\hat{d} + d_\gamma + d_\epsilon \in D$ as D is a convex cone. Now denote $\mu = \hat{\mu} - \gamma - \epsilon$, and then $\hat{\mu} = \mu + \gamma + \epsilon > \mu + \epsilon$ in contradiction to the above. Hence, \hat{y} is weakly ϵ -nondominated for the MOP. \square

Corollary 3.7 Given the MOP, $D \subset \mathbb{R}^m$ a convex cone and $\epsilon \in D$, let $\delta \in \{\delta : \epsilon - \delta v \in D\}$.

(i) If $\hat{x} \in X$ is strictly ϵ -optimal for the $PS(r, v)$, then \hat{x} is ϵ -efficient for the MOP.

(ii) If $\hat{x} \in X$ is ϵ -optimal for the $PS(r, v)$, then \hat{x} is weakly ϵ -efficient for the MOP.

4 Computation of ε -efficient solutions

Based on the results derived in the previous section, we now propose a generic procedure for computing ε -efficient decisions. The main idea is that solving any previous SOP for a (strictly) ϵ -optimal solution produces a (weakly) ε -efficient decision for the MOP, where the exact relationships are summarized in Table 1. Note that the weighted-sum, Guddat and Benson methods give the strongest results among the SOPs discussed.

Table 1: Relationships between ϵ -optimality and ε -efficiency

scalarization method	strict ϵ -opt. \Rightarrow ε -eff.	ϵ -opt. \Rightarrow weak ε -eff.	ϵ -opt. \Rightarrow ε -eff.
weighted-sum	yes	yes	yes
constrained-objective	yes	yes	no
Guddat	yes	yes	yes
Benson	yes	yes	yes
min-max	yes	yes	no
Tchebycheff-norm	yes	yes	no
Pascoletti-Serafini	yes	yes	no

We derived that the Benson method can be considered a special case of the Guddat method. Furthermore, it is well known that the weighted-sum method in general fails for nonconvex problems. Therefore, we recommend that the weighted-sum method be applied for convex problems and the Guddat scalarization be applied for nonconvex problems.

The procedure assumes that based on prior knowledge and experience, the decision maker is able to choose a suitable vector parameter ε and the scalarization parameter $\pi \in \Pi$ required by the SOP of interest. Clearly, these choices depend on the concrete problem to be solved, for which more specific advice can be given. However, note that except for the Pascoletti-Serafini scalarization, each SOP implies an upper bound on the set of allowable values for $\epsilon \in \mathbb{R}$ such that an ϵ -optimal solution for the SOP is guaranteed to be (weakly) ε -efficient for the MOP. For the Pascoletti-Serafini scalarization, this upper bound can be found for the particular case of D being the Pareto cone, $D = \mathbb{R}_{\geq}^m$, and $v \geq 0$. Given that $\epsilon \in \{\delta : \varepsilon - \delta v \in \mathbb{R}_{\geq}^m\}$, we obtain

$$\begin{aligned}
\epsilon &= \max\{\delta : \varepsilon - \delta v \in \mathbb{R}_{\geq}^m\} \\
&= \max\{\delta : \varepsilon \geq \delta v\} \\
&= \max\{\delta : \varepsilon_i \geq \delta v_i \text{ for all } i = 1, \dots, m\} \\
&= \max\{\delta : \varepsilon_i/v_i \geq \delta \text{ for all } i = 1, \dots, m\} \\
&= \min_{i=1, \dots, m} \{\varepsilon_i/v_i\}
\end{aligned}$$

When solving an SOP for ϵ -optimality, we recommend that the actual ϵ should be chosen as the upper bound if the largest tolerance is expected. However, the decision maker may choose any other value of ϵ smaller than the upper bound and hence gain control over the ϵ -efficiency of feasible decisions for the MOP.

The proposed procedure is given in Table 2.

Table 2: Procedure for the computation of ϵ -efficient solutions

- 1. Choice of ε :** Choose an admissible vector $\varepsilon \in \mathbb{R}^m$.
- 2. Choice of the SOP:** Choose a scalarization method SOP.
- 3. Choice of π :** Choose the scalarization parameter $\pi \in \Pi$.
- 4. Computation of ϵ :** Compute the corresponding $\epsilon \in \mathbb{R}$.
- 5. Solving the SOP:** Solve the SOP for an ϵ -optimal solution.

Table 3 summarizes the required parameters $\pi \in \Pi$, the admissible $\varepsilon \in \mathbb{R}^m$ and the corresponding upper bounds of $\epsilon \in \mathbb{R}$ for the SOPs examined in the previous section. Note that in case of the min-max scalarization, the third step (choice of π) does not apply.

Table 3: Parameters for the computation of ϵ -efficient solutions

scalarization method	parameter $\pi \in \Pi$	admissible ε	$\epsilon = \epsilon(\varepsilon, \pi)$
weighted-sum	$w \in \mathbb{R}_{\geq}^m$	$\varepsilon \in \mathbb{R}^m$	$\epsilon = \sum_{i=1}^m w_i \varepsilon_i$
constrained-objective	$\delta \in \mathbb{R}_{\geq}^{m-1}$	$\varepsilon \in \mathbb{R}_{\geq}^m$	$\epsilon = \varepsilon_1$
Guddat	$(x^\circ, w) \in X \times \mathbb{R}_{\geq}^m$	$\varepsilon \in \mathbb{R}_{\geq}^m$	$\epsilon = \sum_{i=1}^m w_i \varepsilon_i$
Benson	$x^\circ \in X$	$\varepsilon \in \mathbb{R}_{\geq}^m$	$\epsilon = \sum_{i=1}^m \varepsilon_i$
min-max	no parameter	$\varepsilon \in \mathbb{R}^m$	$\epsilon = \min_{i=1, \dots, m} \{\varepsilon_i\}$
Tchebycheff-norm	$(r, w) \in \mathbb{R}^m \times \mathbb{R}_{\geq}^m$	$\varepsilon \in \mathbb{R}^m$	$\epsilon = \min_{i=1, \dots, m} \{w_i \varepsilon_i\}$
Pascoletti-Serafini	$(r, v) \in \mathbb{R}^m \times \mathbb{R}_{\geq}^m$	$\varepsilon \in \mathbb{R}_{\geq}^m$	$\epsilon = \min_{i=1, \dots, m} \{\varepsilon_i / v_i\}$

5 Example and Applications

We illustrate the procedure derived in the previous section on the engineering example of designing a four-bar plane truss as given in Figure 1, thereby adopting the model given in Stadler and Dauer (1992) and further discussed in Coello et al. (2002).

The problem is formulated as a biobjective program with the two conflicting objectives of minimizing both the volume V of the truss (f_1) and the displacement Δ of the joint (f_2), subject

to given physical restrictions regarding the feasible cross sectional areas x_1, x_2, x_3, x_4 of the four bars. The stress on the truss is caused by three forces of magnitudes F and $2F$ as depicted in Figure 1. The length L of each bar and the elasticity constants E and σ of the materials involved are modeled as constants.

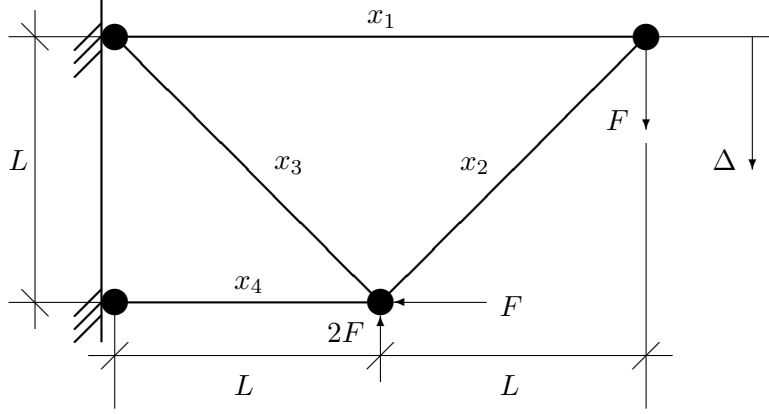


Figure 1: A four-bar plane truss.

The mathematical formulation of this problem is given by

$$\begin{aligned}
 & \text{minimize} \quad \left\{ f_1(x) = L(2x_1 + \sqrt{2}x_2 + \sqrt{2}x_3 + x_4), \right. \\
 & \quad \quad \quad \left. f_2(x) = \frac{FL}{E} \left(\frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} - \frac{2\sqrt{2}}{x_3} + \frac{1}{x_4} \right) \right\} \\
 & \text{subject to} \quad (F/\sigma) \leq x_1 \leq 3(F/\sigma), \\
 & \quad \quad \quad \sqrt{2}(F/\sigma) \leq x_2 \leq 3(F/\sigma), \\
 & \quad \quad \quad \sqrt{2}(F/\sigma) \leq x_3 \leq 3(F/\sigma), \\
 & \quad \quad \quad (F/\sigma) \leq x_4 \leq 3(F/\sigma).
 \end{aligned}$$

Here the constant parameters are chosen as $F = 10$ kN, $E = 2 \times 10^5$ kN/cm², $L = 200$ cm and $\sigma = 10$ kN/cm². Since this problem is convex, we chose the weighted-sum method and initially solved the problem for five hundred weighting parameters $w = (w_1, w_2)$, yielding five hundred Pareto outcomes on the Pareto curve.

Based on the magnitudes of the found values for truss volume (between 1200 cm³ and 2100 cm³) and joint displacement (between 0.01 cm and 0.04 cm), we then allowed a tolerance of $\varepsilon_1 = 50$ cm³ (200 cm³) for the volume and of $\varepsilon_2 = 0.0005$ cm (0.002 cm) for the displacement. Again using the weighted-sum method with same weighting parameters w as before, we solved the scalarized problem while allowing a maximal deviation of $\epsilon = w_1\varepsilon_1 + w_2\varepsilon_2$ from the Pareto optimum as a stopping criterion. Both the Pareto curve and the epsilon-Pareto outcomes obtained in each of the

two cases are given in Figure 2.

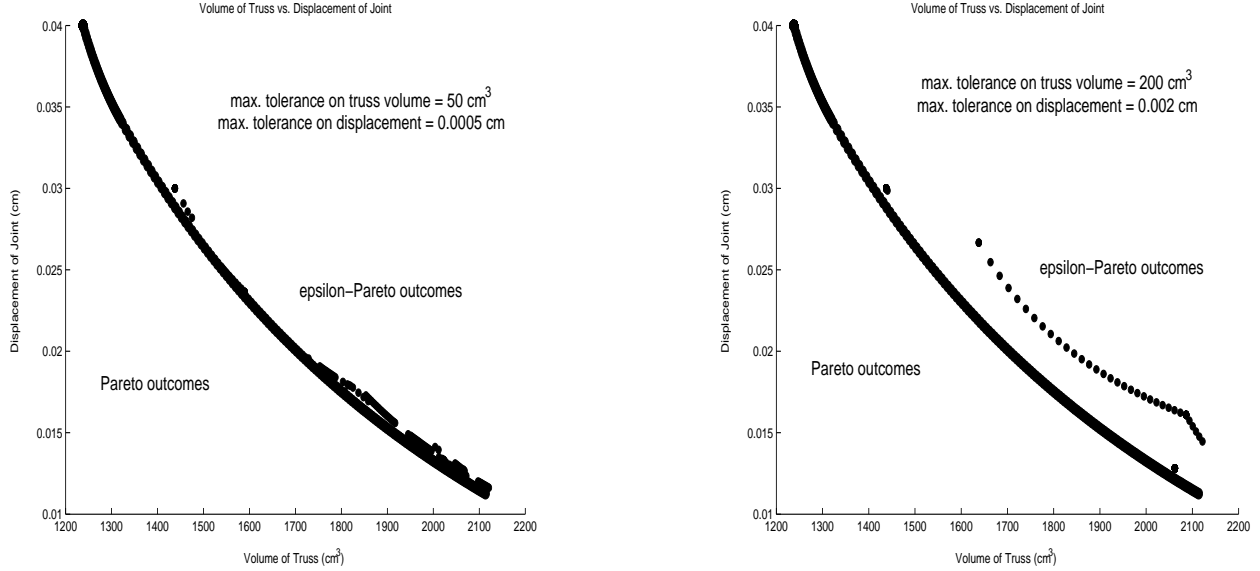


Figure 2: Pareto and epsilon-Pareto outcomes

As we expect, increasing the specified tolerances on truss volume and joint displacement from comparably small to comparably large values causes the ε -Pareto outcomes to move further from the Pareto curve. We also observe that the quality of ε -Pareto outcomes varies significantly along the Pareto curve. The best results are obtained for a truss volume between 1500 cm³ and 2100 cm³ and a joint displacement between 0.015 cm and 0.025 cm. Moreover, the number of distinct outcomes generated is much greater for the Pareto than for the ε -Pareto outcomes.

While the knowledge of efficient decisions and Pareto outcomes has proven to be imperative for modern decision making, approximate efficient solutions (or Pareto outcomes) have not been applied in real-life decision making. We however believe that the capability of computing approximate efficient solutions may enhance and facilitate a decision making process performed within a multiobjective programming-based framework. We envision two types of circumstances in which approximate solutions should be relevant.

First, robustness and sensitivity of efficient solutions and Pareto outcomes can be differently approached. In the classical approach, an efficient decision x is perturbed by δx to observe the corresponding change δy in the outcome y and the ratio $\delta x/\delta y$ is analyzed. In the new approach, perturbation could take place in the objective rather than decision space. A Pareto outcome y could be perturbed by $\delta y = \varepsilon$ to produce an ε -Pareto outcome and the related efficient and ε -efficient decisions could be generated allowing thereby for the analysis of the same ratio but in a different setting.

The second direction is related to the relaxation of Pareto-optimality to ε -Pareto optimality so that the set of Pareto outcomes is enlarged with suboptimal outcomes. Such relaxation can

be useful when dealing with a collection of MOPs for all of which an efficient decision may not exist. Here the MOPs yield a family of Pareto sets and outcomes that are Pareto for one MOP but may not be Pareto for another. This lack of Pareto-optimality can be conveniently modeled and controlled with the concept of ε -Pareto optimality, and the decision maker may gain control over what it takes for a feasible decision to become ε -efficient for all MOPs. In this way, the decision maker can be more closely engaged in the decision process not only at the level of choosing from among efficient solutions but also at the level of deciding solutions' efficiency.

6 Conclusion

Seven scalarization methods are examined with respect to their capability of generating ε -efficient solutions for multiobjective programming problems (MOPs). It is found that for each method, by appropriate choices of the involved parameters, ϵ -optimal solutions for the associated single objective program become ε -efficient for the MOP. Quantitative relationships between ε and ϵ are derived. An example from engineering design illustrates the proposed procedure for computing ε -efficient solutions. Real-life decision making situations benefiting from ε -efficiency are described and their resolution appears to be a promising direction of further research.

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